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Muriel James
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Chapter 1: Functions

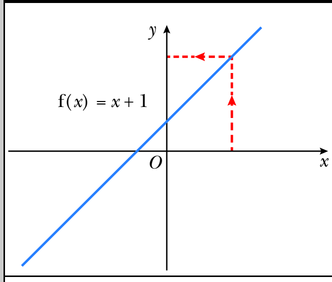
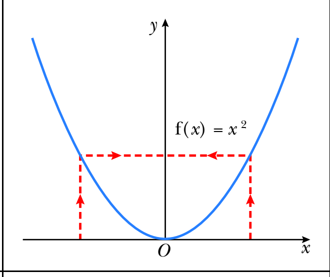
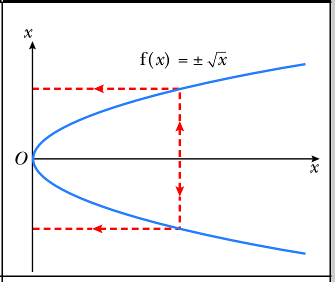
This section will show you how to:

- understand and use the terms: function, domain, range (image set), one-one function, inverse function and composition of functions
- use the notation $f(x) = 2x^3 + 5$, $f: x \mapsto 5x - 3$, $f^{-1}(x)$ and $f^2(x)$
- understand the relationship between $y = f(x)$ and $y = |f(x)|$
- solve graphically or algebraically equations of the type $|ax + b| = c$ and $|ax + b| = cx + d$
- explain in words why a given function is a function or why it does not have an inverse
- find the inverse of a one-one function and form composite functions
- use sketch graphs to show the relationship between a function and its inverse.

1.1 Mappings

REMINDER

The table below shows one-one, many-one and one-many mappings.

one-one	many-one	one-many
		
For one input value there is just one output value.	For two input values there is one output value.	For two input value there are two output values.

Exercise 1.1

Determine whether each of these mappings is one-one, many-one or one-many.

1 $x \mapsto 2x + 3 \quad x \in \mathbb{R}$

2 $x \mapsto x^2 + 4 \quad x \in \mathbb{R}$

3 $x \mapsto 2x^3 \quad x \in \mathbb{R}$

4 $x \mapsto 3^x \quad x \in \mathbb{R}$

5 $x \mapsto \dfrac{-1}{x} \quad x \in \mathbb{R},\ x > 0$

6 $x \mapsto x^2 + 1 \quad x \in \mathbb{R},\ x \geqslant 0$

7 $x \mapsto \dfrac{2}{x} \quad x \in \mathbb{R},\ x > 0$

8 $x \mapsto \pm\sqrt{x} \quad x \in \mathbb{R},\ x \geqslant 0$

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1.2 Definition of a function

REMINDER

A function is a rule that maps each *x* value to just one *y* value for a defined set of input values.

This means that mappings that are either {one-one, many-one} are called functions.

The mapping $x \mapsto x + 1$ where $x \in \mathbb{R}$, is a one-one function.

The function can be defined as $\mathrm{f} : x \mapsto x + 1, \quad x \in \mathbb{R}$ or $\mathrm{f}(x) = x + 1, \quad x \in \mathbb{R}$.

The set of input values for a function is called the **domain** of the function.

The set of output values for a function is called the **range** (or image set) of the function.

WORKED EXAMPLE 1

The function f is defined by $\mathrm{f}(x) = (x-1)^2 + 4$ for $0 \leqslant x \leqslant 5$.

Find the range of f.

Answers

$\mathrm{f}(x) = (x-1)^2 + 4$ is a positive quadratic function so the graph will be of the form

$(x-1)^2 + 4$

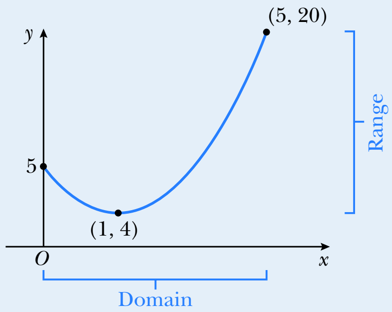
This part of the expression is a square so it will always be $\geqslant 0$.
The smallest value it can be is 0. This occurs when $x = 1$.

The minimum value of the expression is $0 + 4 = 4$ and this minimum occurs when $x = 1$.

So the function $\mathrm{f}(x) = (x-1)^2 + 4$ will have a minimum point at the point $(1, 4)$.

When $x = 0$, $y = (0-1)^2 + 4 = 5$.

When $x = 5$, $y = (5-1)^2 + 4 = 20$.



The range is $1 \leqslant \mathrm{f}(x) \leqslant 20$.

Exercise 1.2

1 Which of the mappings in **Exercise 2.1** are functions?

2 Find the range for each of these functions.

a $\mathrm{f}(x) = x - 9, \quad -2 \leqslant x \leqslant 8$

b $\mathrm{f}(x) = 2x - 2, \quad 0 \leqslant x \leqslant 6$

c $\mathrm{f}(x) = 7 - 2x, \quad -3 \leqslant x \leqslant 5$

d $\mathrm{f}(x) = 2x^2, \quad -4 \leqslant x \leqslant 3$

e $\mathrm{f}(x) = 3^x, \quad -4 \leqslant x \leqslant 3$

f $\mathrm{f}(x) = \dfrac{-1}{x}, \quad 1 \leqslant x \leqslant 6$

3 The function g is defined as $\mathrm{g}(x) = x^2 - 5$ for $x \geqslant 0$.

Find the range of g.

4 The function f is defined by $\mathrm{f}(x) = 4 - x^2$ for $x \in \mathbb{R}$.

Find the range of f.

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5 The function f is defined by $f(x) = 3 - (x - 1)^2$ for $x \geqslant 1$.
Find the range of f.

6 The function f is defined by $f(x) = (4x + 1)^2 - 2$ for $x \geqslant -\frac{1}{4}$.
Find the range of f.

7 The function f is defined by $f : x \mapsto 8 - (x - 3)^2$ for $2 \leqslant x \leqslant 7$.
Find the range of f.

8 The function f is defined by $f(x) = 3 - \sqrt{x - 1}$ for $x \geqslant 1$.
Find the range of f.

9 Find the largest possible domain for the following functions.

a $f(x) = \frac{1}{x + 3}$ **b** $f(x) = \frac{3}{x - 2}$ **c** $\frac{4}{(x - 3)(x + 2)}$

d $f(x) = \frac{1}{x^2 - 4}$ **e** $f : x \mapsto \sqrt{x^3 - 4}$ **f** $f : x \mapsto \sqrt{x + 5}$

g $g : x \mapsto \frac{1}{\sqrt{x - 2}}$ **h** $f : x \mapsto \frac{x}{\sqrt{3 - 3x}}$ **i** $f : x \mapsto 1 - x^2$

1.3 Composite functions

REMINDER

- When one function is followed by another function, the resulting function is called a **composite function.**
- $fg(x)$ means the function g acts on x first, then f acts on the result.
- $f^2(x)$ means $ff(x)$, so you apply the function f twice.

WORKED EXAMPLE 2

$f : x \mapsto 4x + 3$ for $x \in \mathbb{R}$
$g : x \mapsto 2x^2 - 5$ for $x \in \mathbb{R}$
Find fg(3).

Answer

fg(3) — g acts on 3 first and $g(3) = 2 \times 3^2 - 5 = 13$.
$= f(13)$
$= 4 \times 13 + 3$
$= 55$

WORKED EXAMPLE 3

$g(x) = 2x^2 - 2$ for $x \in \mathbb{R}$
$h(x) = 4 - 3x$ for $x \in \mathbb{R}$
Solve the equation $hg(x) = -14$.

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Answers

$\text{hg}(x)$	g acts on x first and $\text{g}(x) = 2x^2 - 2$.
$= \text{h}(2x^2 - 2)$	h is the function 'triple and take from 4'.
$= 4 - 3(2x^2 - 2)$	Expand the brackets.
$= 4 - 6x^2 + 6$	
$= 10 - 6x^2$	
$\text{hg}(x) = -14$	
$-14 = 10 - 6x^2$	Set up and solve the equation.
$24 = 6x^2$	
$4 = x^2$	
$x = \pm 2$	

Exercise 1.3

1 $\text{f}(x) = 2 - x^2$ for $x \in \mathbb{R}$

$\text{g}(x) = \frac{x}{2} + 3$ for $x \in \mathbb{R}$

Find the value of $\text{gf}(4)$.

2 $\text{f}(x) = (x-2)^2 - 2$ for $x \in \mathbb{R}$

Find $\text{f}^2(3)$.

3 The function f is defined by $\text{f}(x) = 1 + \sqrt{x-3}$ for $x \geqslant 3$.

The function g is defined by $\text{g}(x) = \frac{-3}{x} - 1$ for $x > 0$.

Find gf(7).

4 The function f is defined by $\text{f}(x) = (x-2)^2 + 3$ for $x > -2$.

The function g is defined by $\text{g}(x) = \frac{3x+4}{x+2}$ for $x > 2$.

Find fg(6).

5 $\text{f} : x \mapsto 3x - 1$ for $x > 0$

$\text{g} : x \mapsto \sqrt{x}$ for $x > 0$

Express each of the following in terms of f and g.

a $x \mapsto 3\sqrt{x} - 1$ **b** $x \mapsto \sqrt{3x-1}$

6 The function f is defined by $\text{f} : x \mapsto 2x - 1$ for $x \in \mathbb{R}$.

The function g is defined by $\text{g} : x \mapsto \frac{8}{4-x}$ for $x \neq 4$.

Solve the equation $\text{gf}(x) = 5$.

7 $\text{f}(x) = 2x^2 + 3$ for $x > 0$

$\text{g}(x) = \frac{5}{x}$ for $x > 0$

Solve the equation $\text{fg}(x) = 4$.

8 The function f is defined, for $x \in \mathbb{R}$, by $\text{f} : x \mapsto \frac{2x-1}{x-3}$, $x \neq 3$.

The function g is defined, for $x \in \mathbb{R}$, by $\text{g} : x \mapsto \frac{x+1}{2} x \neq 1$.

Solve the equation $\text{fg}(x) = 4$.

TIP
Before writing your final answers, compare your solutions with the domains of the original functions.

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9 The function g is defined by $g(x) = 1 - 2x^2$ for $x \geqslant 0$.
The function h is defined by $h(x) = 3x - 1$ for $x \geqslant 0$.
Solve the equation $gh(x) = -3$ giving your answer(s) as exact value(s).

10 The function f is defined by $f : x \mapsto x^2$ for $x \in \mathbb{R}$.
The function g is defined by $g : x \mapsto x + 2$ for $x \in \mathbb{R}$.
Express each of the following as a composite function, using only f and g.

a $x \mapsto (x+2)^2$ **b** $x \mapsto x^2 + 2$ **c** $x \mapsto x + 4$ **d** $x \mapsto x^4$

11 The functions f and g are defined for $x > 0$ by $f : x \mapsto x + 3$ and $g : x \mapsto \sqrt{x}$
Express in terms of f and g

a $x \mapsto \sqrt{x+3}$ **b** $x \mapsto x + 6$ **c** $x \mapsto \sqrt{x} + 3$

12 Given the functions $f(x) = \sqrt{x}$ and $g(x) = \dfrac{x-5}{2x+1}$,

a Find the domain and range of g.

b Solve the equation $g(x) = 0$.

c Find the domain and range of fg.

1.4 Modulus functions

REMINDER

- The **modulus** (or **absolute value**) of a number is the magnitude of the number without a sign attached.
- **The modulus of** x, written as $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

- **The statement** $|x|, = k$, where $\geqslant 0$, means that $x = k$ or $x = -k$.

WORKED EXAMPLE 4

a $|4x + 3| = x + 18$ **b** $|2x^2 - 9| = 7$

Answers

a $|4x + 3| = x + 18$

$4x + 3 = x + 18$ or $4x + 3 = -x - 18$

$3x = 15$ $\quad 5x = -21$

$x = 5$ $\quad x = -\dfrac{21}{5}$

Solution is : $x = 5$ or $-\dfrac{21}{5}$

b $|2x^2 - 7| = 9$

$2x^2 - 7 = 9$ or $2x^2 - 7 = -9$

$2x^2 = 16$ $\quad 2x^2 = -2$

$x^2 = 8$ $\quad x^2 = -1$

$x = \pm 2\sqrt{2}$

Solution is : $x = \pm 2\sqrt{2}$

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Exercise 1.4

> **TIP**
> Remember to check your answers to make sure that they satisfy the original equation.

1 Solve.

a $|2x-1| = 11$ **b** $|2x+4| = 8$ **c** $|6-3x| = 4$

d $\left|\frac{x-2}{5}\right| = 6$ **e** $\left|\frac{3x+4}{3}\right| = 4$ **f** $\left|\frac{9-2x}{3}\right| = 4$

g $\left|\frac{x}{3}-6\right| = 1$ **h** $\left|\frac{2x+5}{3}+\frac{2x}{5}\right| = 3$ **i** $|2x-6| = x$

2 Solve.

a $\left|\frac{2x-5}{x+4}\right| = 3$ **b** $\left|\frac{4x+2}{x+3}\right| = 3$ **c** $\left|1+\frac{2x+5}{x+3}\right| = 4$

d $|2x-3| = 3x$ **e** $2x+|3x-4| = 5$ **f** $7-|1-2x| = 3x$

3 Solve giving your answers as exact values if appropriate.

a $|x^2-4| = 5$ **b** $|x^2+5| = 11$ **c** $|9-x^2| = 3-x$

d $|x^2-3x| = 2x$ **e** $|x^2-16| = 2x+1$ **f** $|2x^2-1| = x+2$

g $|3-2x^2| = x$ **h** $|x^2-4x| = 3-2x$ **i** $|2x^2-2x+5| = 1-x$

4 Solve each of the following pairs of simultaneous equations.

a $y = x+4$
$y = |x^2-2|$

b $y = 1-x$
$y = |4x^2-4x|$

1.5 Graphs of $y = |f(x)|$ where $f(x)$ is linear

Exercise 1.5

1 Sketch the graphs of each of the following functions showing the coordinates of the points where the graph meets the axes.

a $y = |x-2|$ **b** $y = |3x-3|$ **c** $y = |3-x|$

d $y = \left|\frac{1}{3}x-3\right|$ **e** $y = |6-3x|$ **f** $y = \left|5-\frac{1}{2}x\right|$

2 a Complete the table of values for $y = 3-|x-1|$.

x	–2	–1	0	1	2	3	4
y		1		3			

b Draw the graph of $y = 3-|x-1|$ for $-2 \leqslant x \leqslant 4$.

3 Draw the graphs of each of the following functions.

a $y = |2x|+2$ **b** $y = |x|-2$ **c** $y = 4-|3x|$

d $y = |x-1|+3$ **e** $y = |3x-6|-2$ **f** $y = 4-\left|\frac{1}{2}x\right|$

4 Given that each of these functions is defined for the domain $-3 \leqslant x \leqslant 4$, find the range of

a $f: x \mapsto 6-3x$ **b** $g: x \mapsto |6-3x|$ **c** $h: x \mapsto 6-|3x|$.

5 **a** $\text{f} : x \mapsto 2 - 2x$ for $-1 \leqslant x \leqslant 5$

b $\text{g} : x \mapsto |2 - 2x|$ for $-1 \leqslant x \leqslant 5$

c $\text{h} : x \mapsto 2 - |2x|$ for $-1 \leqslant x \leqslant 5$

Find the range of each function for $-1 \leqslant x \leqslant 5$.

6 **a** Sketch the graph of $y = |3x - 2|$ for $-4 < x < 4$, showing the coordinates of the points where the graph meets the axes.

b On the same diagram, sketch the graph of $y = x + 3$.

c Solve the equation $|3x - 2| = x + 3$.

7 A function f is defined by $\text{f}(x) = 2 - |3x - 1|$, for $-1 \leqslant x \leqslant 3$.

a Sketch the graph of $y = \text{f}(x)$.

b State the range of f.

c Solve the equation $\text{f}(x) = -2$.

8 **a** Sketch on a single diagram, the graphs of $x + 3y = 6$ and $y = |x + 2|$.

b Solve the inequality $|x + 2| < \frac{1}{3}(6 - x)$.

1.6 Inverse functions

REMINDER

- **The inverse of the function** $\text{f}(x)$ is written as $\text{f}^{-1}(x)$.
- The domain of $\text{f}^{-1}(x)$ is the range of $\text{f}(x)$.
- The range of $\text{f}^{-1}(x)$ is the domain of $\text{f}(x)$.
- It is important to remember that not every function has an inverse.
- An inverse function $\text{f}^{-1}(x)$ can exist if, and only if, the function $\text{f}(x)$ is a one-one mapping.

WORKED EXAMPLE 5

$\text{f}(x) = (x + 3)^2 - 1$ for $x > -3$

a Find an expression for $\text{f}^{-1}(x)$.

b Solve the equation $\text{f}^{-1}(x) = 3$.

Answers

a $\text{f}(x) = (x + 3)^2 - 1$ for $x > -3$

Step 1: Write the function as $y =$ → $y = (x + 3)^2 - 1$

Step 2: Interchange the x and y variables. → $x = (y + 3)^2 - 1$

Step 3: Rearrange to make y the subject. → $x + 1 = (y + 3)^2$

$$\sqrt{x + 1} = y + 3$$
$$y = \sqrt{x + 1} - 3$$

$\text{f}^{-1}(x) = \sqrt{x + 1} - 3$

b $\text{f}^{-1}(x) = 3$.

$$\sqrt{x + 1} - 3 = 3$$
$$\sqrt{x + 1} = 6$$
$$x + 1 = 36$$
$$x = 35$$

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Exercise 1.6

1 $f(x) = (x+2)^2 - 3$ for $x \geqslant -2$.
Find an expression for $f^{-1}(x)$.

2 $f(x) = \dfrac{5}{x-2}$ for $x \geqslant 0$.
Find an expression for $f^{-1}(x)$.

3 $f(x) = (3x-2)^2 + 3$ for $x \geqslant \dfrac{2}{3}$.
Find an expression for $f^{-1}(x)$.

4 $f(x) = 4 - \sqrt{x-2}$ for $x \geqslant 2$.
Find an expression for $f^{-1}(x)$.

5 $f : x \mapsto 3x - 4$ for $x > 0$ $\quad g : x \mapsto \dfrac{4}{4-x}$ for $x \neq 4$.
Express $f^{-1}(x)$ and $g^{-1}(x)$ in terms of x.

6 $f(x) = (x-2)^2 + 3$ for $x > 2$
a Find an expression for $f^{-1}(x)$.
b Solve the equation $f^{-1}(x) = f(4)$.

7 $g(x) = \dfrac{3x+1}{x-3}$ for $x > 3$
a Find an expressions for $g^{-1}(x)$ and comment on your result.
b Solve the equation $g^{-1}(x) = 6$.

8 $f(x) = \dfrac{x}{2} - 2$ for $x \in \mathbb{R}$ $\quad g(x) = x^2 - 4x$ for $x \in \mathbb{R}$
a Find $f^{-1}(x)$.
b Solve $fg(x) = f^{-1}(x)$ leaving answers as exact values.

9 $f : x \mapsto \dfrac{3x+1}{x-1}$ for $x \neq 1$ $\quad g : x \mapsto \dfrac{x-2}{3}$ for $x > -2$
Solve the equation $f(x) = g^{-1}(x)$.

10 If $f(x) = \dfrac{x^2-9}{x^2+4}$ $x \in \mathbb{R}$ find an expression for $f^{-1}(x)$.

11 If $f(x) = 2\sqrt{x}$ and $g(x) = 5x$, solve the equation $f^{-1}g(x) = 0.01$.

12 Find the value of the constant k such that $f(x) = \dfrac{2x-4}{x+k}$ is a self-inverse function.

TIP
A self-inverse function is one for which $f(x) = f^{-1}(x)$, for all values of x in the domain.

13 The function f is defined by $f(x) = x^3$. Find an expression for $g(x)$ in terms of x for each of the following:
a $fg(x) = 3x + 2$
b $gf(x) = 3x + 2$

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14 Given $f(x) = 2x + 1$ and $g(x) = \frac{x+1}{2}$ find the following.

a f^{-1} **b** g^{-1} **c** $(fg)^{-1}$ **d** $(gf)^{-1}$ **e** $f^{-1}g^{-1}$ **f** $g^{-1}f^{-1}$

Write down any observations from your results.

15 Given that $fg(x) = \frac{x+2}{3}$ and $g(x) = 2x + 5$ find $f(x)$.

16 Functions f and g are defined for all real numbers.
$g(x) = x^2 + 7$ and $gf(x) = 9x^2 + 6x + 8$. Find $f(x)$.

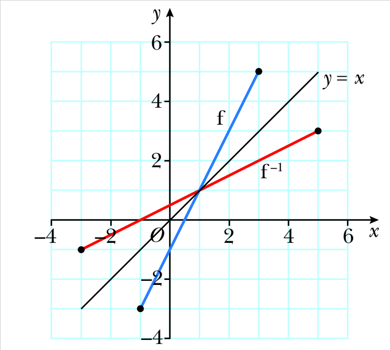
1.7 The graph of a function and its inverse

REMINDER

The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

This is true for all one-one functions and their inverse functions.

This is because: $ff^{-1}(x) = x = f^{-1}f(x)$.



Some functions are called **self-inverse functions** because f and its inverse f^{-1} are the same.

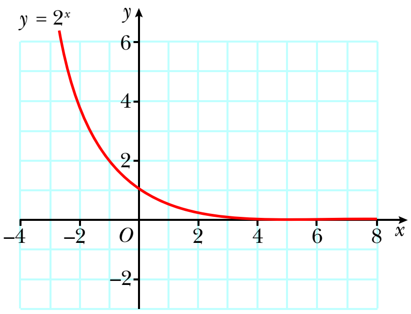
If $f(x) = \frac{1}{x}$ for $x \neq 0$, then $f^{-1}(x) = \frac{1}{x}$ for $x \neq 0$.

So $f(x) = \frac{1}{x}$ for $x \neq 0$ is an example of a self-inverse function.

When a function f is self-inverse, the graph of f will be symmetrical about the line $y = x$.

Exercise 1.7

1 On a copy of the grid, draw the graph of the inverse of the function $y = 2^{-x}$.



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2 $f(x) = x^2 + 5, x \geqslant 0$.

On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing the coordinates of any points where the curves meet the coordinate axes.

3 $g(x) = \frac{1}{2}x^2 - 4$ for $x \geqslant 0$.

Sketch, on a single diagram, the graphs of $y = g(x)$ and $y = g^{-1}(x)$, showing the coordinates of any points where the curves meet the coordinate axes.

4 The function f is defined by $f(x) = 3x - 6$ for all real values of x

a Find the inverse function $f^{-1}(x)$.

b Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

c Write down the point of intersection of the graphs $f(x)$ and $f^{-1}(x)$.

5 Given the function $f(x) = x^2 - 2x$ for $x \geqslant 1$.

a Explain why $f^{-1}(x)$ exists and find $f^{-1}(x)$.

b State the range of the function $f^{-1}(x)$.

c Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

d Write down where $f^{-1}(x)$ crosses the y axis.

6 **a** By finding $f^{-1}(x)$ show that $f(x) = \dfrac{3x - 1}{2x - 3}$ $x \in \mathbb{R}$, $x \neq \dfrac{3}{2}$ is a self-inverse function.

b Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

c Write down the coordinates of the intersection of the graphs with the coordinate axes.

Summary

Functions

A function is a rule that maps each x-value to just one y-value for a defined set of input values.

Mappings that are either $\left\{\begin{array}{l}\text{one-one}\\ \text{many-one}\end{array}\right.$ are called functions.

The set of input values for a function is called the **domain** of the function.

The set of output values for a function is called the **range** (or image set) of the function.

Modulus function

The modulus of x, written as $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Composite functions

$fg(x)$ means the function g acts on x first, then f acts on the result.

$f^2(x)$ means $ff(x)$.